A Category Theoretic View of Nondeterministic Recursive Program Schemes

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Recursive program schemes

- capture the structure of functional programs and
- serve to give a semantics for such programs.
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<th>recursive program schemes (RPS)</th>
<th>nondeterministic recursive program schemes (NDRPS)</th>
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<td>classical</td>
<td>1970s: Courcelle, Nivat,</td>
<td>~1980: Nivat, Arnold, Boudol, Poigné</td>
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<td>Guessarian, ...</td>
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<td>category</td>
<td>[De Marchi Ghani Lüth 2003]</td>
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<td>theoretical</td>
<td>[Milius Moss 2006]</td>
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1. Nondeterministic Recursive Program Schemes
2. An Uninterpreted Category Theoretic Semantics
3. Comparison with Related Work
4. Future Work
Definition (classical RPS)

- disjoint finite sets $G$ – given operation symbols
  \[ \Phi \] – new operation symbols
  \[ X \] – variables

\[ \phi(x_1, \ldots, x_n) = t^\phi(x_1, \ldots, x_n) \quad \text{for all } \phi \in \Phi, \]
\[ t^\phi \text{ term in } G \cup \Phi \]
Classical NDRPSs

Definition (classical NDRPS)

- disjoint finite sets $G$ – given operation symbols
  $\Phi$ – new operation symbols
  $X$ – variables
- special binary operation symbol or $\notin G \cup \Phi$
- $\phi(x_1, \ldots, x_n) = t^\phi(x_1, \ldots, x_n)$ for all $\phi \in \Phi_n$, $t^\phi$ term in $G \cup \Phi \cup \{\text{or}\}$

Example

$\phi(x) = g(x, x) \text{ or } g(x, \phi(x))$
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\[ \phi(x) = g(x, x) \text{ or } g(x, \phi(x)) \]
Definition (category theoretic NDRPS)

- polynomial endofunctors $H$ and $V$ on $\textbf{Set}$
- natural transformation $e : V \rightarrow \mathcal{P}^+ F^{H+V}$

Example (continued)

$H_X = X \times X \circ g, \quad V_X = X \circ \phi$

$e_X : V_X \rightarrow \mathcal{P}^+ F^{H+V}$
given by $e_X(\phi(x)) = \{ g(x, x), g(x, \phi(x)) \}$
Definition (category theoretic NDRPS)

- polynomial endofunctors \( H \) and \( V \) on \( \textbf{Set} \)
- natural transformation \( e : V \rightarrow \mathcal{P} + F^{H+V} \)

Example (continued)

\[
HX = \underbrace{X \times X}_{g}, \quad VX = \underbrace{X}_{\phi}
\]

\[
e_X : VX \rightarrow \mathcal{P} + F^{H+V}X \text{ given by } \\
e_X(\phi(x)) = \{ g(x, x), g(x, \phi(x)) \}
\]
Semantics of NDRPSs

- So far: NDRPSs purely syntactical constructs

- Now: give semantics for NDRPSs
  - non-trivial: recursion and non-determinism
  - uninterpreted and interpreted semantics
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- Now: give semantics for NDRPSs
  - non-trivial: recursion and non-determinism
  - uninterpreted and interpreted semantics

- Here: uninterpreted semantics only
  - for classical NDRPSs in [Arnold Nivat 1977]
  - for category theoretic NDRPSs following
Uninterpreted Solutions

- $(\mathcal{P}^+, \eta^+, \mu^+)$ nonempty powerset monad
- $(T^H, \eta^H, \mu^H)$ with $\kappa^H : H \to T^H$ free completely iterative monad over $H$ [Milius 2005]
Uninterpreted Solutions

- \((P^+, \eta^+, \mu^+)\) nonempty powerset monad
- \((T^H, \eta^H, \mu^H)\) with \(\kappa^H : H \to T^H\) free completely iterative monad over \(H\) [Milius 2005]
- distributive law \(H P^+ \to P^+ H\) extends to \(T^H P^+ \to P^+ T^H\)
- \(P^+ T^H\) composite monad by [Beck 1969]
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- \(\mathcal{P}^+T^H\) composite monad by [Beck 1969]

Definition (uninterpreted solution of NDRPS \(e\))

Natural transformation \(e^\dagger : V \to \mathcal{P}^+ T^H\) such that the diagram

\[
\begin{array}{c}
V \\
\downarrow^e \\
\mathcal{P}^+ F^H + V \\
\downarrow \quad \quad \quad \downarrow \quad \quad \quad \quad \quad \downarrow \\
\mathcal{P}^+ \mathcal{P}^+ T^H \\
\end{array}
\]

\[
\begin{array}{c}
\mathcal{P}^+ T^H \\
\downarrow \quad \quad \quad \downarrow \\
\mathcal{P}^+ T^H \\
\end{array}
\]

commutes.
Main Result

Definition

A NDRPS $e : V \rightarrow \mathcal{P}^+ F^H + V$ is called guarded if it factors

$$e \equiv ( V \xrightarrow{e'} \mathcal{P}^+ H F^H + V \xrightarrow{\cdots} \mathcal{P}^+ F^H + V ).$$
Main Result

Definition

A NDRPS $e : V \rightarrow \mathcal{P}^+ F^H + V$ is called **guarded** if it factors

$$ e \equiv ( V \xrightarrow{e'} \mathcal{P}^+ HF^H + V \xrightarrow{...} \mathcal{P}^+ F^H + V ). $$

Theorem

*Every guarded NDRPS has a canonical greatest uninterpreted solution (w. r. t. componentwise and pointwise subset inclusion).*

Moreover, for all uninterpreted solutions of a guarded NDRPS the set of finite cuttings of terms is the same.
Example (continued)

- NDRPS given by $e_X(\phi(x)) = \{ g(x, x), g(x, \phi(x)) \}$
- greatest solution: $e^\dagger_X(\phi(x)) = \{ \}

another solution: remove rightmost tree
An Intermediate Result

From $H$, $\mathcal{P}^+$ and $\lambda : H\mathcal{P}^+ \rightarrow \mathcal{P}^+H$ we obtain

- a functor $\mathcal{H} = H \cdot \_ + \text{Id}$ on $[\text{Set}, \text{Set}]$;
- a monad $\mathcal{M} = (\mathcal{P}^+ \cdot \_, \eta^+\_, \mu^+\_)$ on $[\text{Set}, \text{Set}]$;
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- a monad $\mathcal{M} = (\mathcal{P}^+ \cdot _+, \eta^+_-, \mu^+_-) \text{ on } [\text{Set}, \text{Set}]$;
- a distributive law $\Lambda : \mathcal{H}\mathcal{M} \to \mathcal{M}\mathcal{H}$;
- equivalently, a lifting $\bar{\mathcal{H}}$ of $\mathcal{H}$ to $[\text{Set}, \text{Set}]_\mathcal{M}$ [Mulry 1994].
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**Theorem**

$$\eta^+(HT^H + \text{Id}) \cdot [\mu^H \cdot \kappa^H T^H, \eta^H]^{-1} : T^H \to \mathcal{P}^+(HT^H + \text{Id})$$

is a weakly final $\overline{\mathcal{H}}$-coalgebra.
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**Proof (idea).**

For every $\bar{\mathcal{H}}$-coalgebra, canonically construct a greatest homomorphism into the final $\bar{\mathcal{H}}$-coalgebra using
- category theoretic determinization and
- unique solutions of deterministic recursive equations.
Relation to Classical NDRPSs

Category theoretical NDRPSs cover and extend the classical ones [Arnold Nivat 1977]:

- definition of NDRPS
  - translation classical to category theoretical NDRPS
  - generalization to infinite equation systems and infinite sets

- uninterpreted semantics
  - recover classical greatest fixed point result
  - goes beyond greatest fixed points
Relation to Category Theoretical RPSs

[Milius Moss 2006]

RPS $V \rightarrow T^{H+V}$

NDRPS $V \rightarrow \mathcal{P}+F^{H+V}$

A = Set, $H$ and $V$ polynomial, $F^H+V \hookrightarrow T^H+V$ replace $(P+\eta+\mu)$ by $(\text{Id}, \text{id}, \text{id})$.
Relation to Category Theoretical RPSs

\[ \text{[Milius Moss 2006]} \]

RPS \( V \rightarrow T^{H+V} \)

NDRPS \( V \rightarrow P^+ F^{H+V} \)

\( A = \text{Set}, \) 
H and V polynomial, 
\( F^{H+V} \hookrightarrow T^{H+V} \)

replace \((P^+, \eta^+, \mu^+)\) by \((\text{Id}, \text{id}, \text{id})\)

restricted RPS \( V \rightarrow F^{H+V} \)
Relation to Category Theoretical RPSs

\[ [\text{Milius Moss 2006}] \]
RPS \( V \rightarrow T^{H+V} \)
unique uninterp. solution

\[ A = \text{Set}, \]
\( H \) and \( V \) polynomial,
\( F^{H+V} \leftrightarrow T^{H+V} \)

restricted RPS \( V \rightarrow F^{H+V} \)
unique uninterp. solution

NDRPS \( V \rightarrow \mathcal{P}^+ F^{H+V} \)
canonical greatest uninterp. solution

replace \((\mathcal{P}^+, \eta^+, \mu^+)\) by \((\text{Id}, \text{id}, \text{id})\)
Possibilities due to category theoretic framework:
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- generalize $H$ and $V$ (analytic/weak pullback preserving functors?)

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Future Work

Possibilities due to category theoretic framework:

- generalize $H$ and $V$ (analytic/weak pullback preserving functors?)

- try other effects / monads
  - composite RPSs / $((-)^E, \eta^E, \mu^E)$
  - partial RPSs / $(\text{Id} + 1, \eta^{pa}, \mu^{pa})$
  - NDRPSs with $\emptyset$ / $(\mathcal{P}, \eta^P, \mu^P)$
  - probabilistic RPSs / $(\mathcal{D}, \eta^D, \mu^D)$
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- generalize to infinite terms (using complete Elgot monads?)
- interpreted solutions (using [Milius Palm S 09]?)

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Thank you... for your attention!

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- captures the structure of a functional program
- is a system of recursive equations for operation symbols
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**Example**

$$\phi(x) = f(x, \phi(g(x)))$$
$$\psi(x) = f(\phi(g(x)), g(g(x)))$$
Definition (category theoretic RPS [Milius Moss 2006])

- endofunctors $H$ and $V$ on a category $\mathcal{A}$ with finite coproducts
  s.t. $\forall X \in \text{Obj}(\mathcal{A}) \exists$ final coalgebras for $H(\cdot) + X$ and $(H + V)(\cdot) + X$
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Example

$\mathcal{A} = \text{Set}$, $HX = X \times X + X$, $VX = X + X$

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$e_X(\phi(x)) = f(x, \phi(g(x)))$

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