Abstract. Dana Scott’s model of $\lambda$-calculus was based on a limit construction which started from an algebra of a suitable endofunctor $F$ and continued by iterating $F$. We demonstrate that this is a special case of the concept we call coalgebra relatively terminal w.r.t. the given algebra $A$. This means a coalgebra together with a universal coalgebra-to-algebra morphism into $A$.

We prove that by iterating $F$ countably many times we obtain the relatively terminal coalgebras whenever $F$ preserves limits of $\omega^{op}$-chains. If $F$ is finitary, we need in general $\omega + \omega$ steps. And for arbitrary accessible (=bounded) set functors we need an ordinal number of steps in general. Scott’s result is captured by the fact that in a CPO-enriched category, assuming that $F$ is locally continuous, $\omega$ steps are sufficient for algebras given by projections.

1. Introduction

Terminal coalgebras of endofunctors $F$ play an important role in the theory of systems expressed by $F$-coalgebras. Jan Rutten demonstrated in [R] that the terminal coalgebra is the coalgebra of behaviours of states in such systems. The classical construction (dualizing that of initial algebras in [Ad]) is to form the limit of the $\omega^{op}$-chain

$1 \xleftarrow{\alpha} F1 \xleftarrow{Fa} FF1 \xleftarrow{FFa} \ldots$

where $\alpha : F1 \to 1$ is the (trivial) terminal algebra. Another source of interest in terminal coalgebras stems from the model of untyped $\lambda$-calculus presented by Dana Scott [S]. However, Scott did not use a terminal coalgebra: rather, he used, for a "suitable" algebra $\alpha : FA \to A$, the limit of the analogous $\omega^{op}$-chain

$A \xleftarrow{\alpha} FA \xleftarrow{Fa} FFA \xleftarrow{FFa} \ldots$

The properties of the endofunctor $F$ he used made it clear that $F$ preserves this limit. Whenever this happens, we are going to prove that the limit carries the structure of a coalgebra for which the first projection (into $A$) is a universal coalgebra-to-algebra morphism. This is called a coalgebra relatively terminal to the given algebra. But in general, this limit $F^\omega A = \lim_{i<\omega} F^i A$ carries itself an obvious structure of an algebra $\overline{\alpha} : F(F^\omega A) \to F^\omega A$. We prove that this algebra has always the same relatively terminal coalgebra as the original one.

For finitary set functors relatively terminal coalgebras are always obtained in $\omega + \omega$ steps: we first form the algebra $F^\omega A$ in $\omega$ steps, and then we perform the same construction on it - in the next $\omega$ steps we get a limit preserved by $F$, thus, yielding a relatively terminal coalgebra for $F^\omega A$, consequently, also for $A$. This generalizes the result of James Worell [W] that terminal coalgebras of finitary functors take $\omega + \omega$ steps. Surprisingly, finitary endofunctors of many-sorted sets can require an arbitrarily large number of steps for the terminal algebra.

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All accessible (=bounded) set functors $F$ have relatively terminal coalgebras: if $F$ preserves $\lambda$-filtered colimits, we need $\lambda + \lambda$ steps of iteration. More generally, every monomorphisms preserving, accessible endofunctor of a locally presentable category has relatively terminal coalgebras obtained by the iterative construction. This is a new result even for (absolutely) terminal coalgebras: the proof that a terminal coalgebra exists, presented by Michael Barr [B], was not constructive.

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2. Terminal coalgebras of accessible functors

Before coming to relatively terminal coalgebras, we formulate a result concerning terminal coalgebras: if an accessible functor preserves monomorphisms, then it has a terminal coalgebra obtained by the terminal chain. Recall that a functor is called finitary if it preserves filtered colimits, and $\lambda$-accessible if it preserves $\lambda$-filtered colimits for an infinite cardinal $\lambda$.

2.1. Notation. Let $F$ be an endofunctor of a complete category. We denote by

$$F^i1 \quad (i \in \text{Ord})$$

the terminal chain with connecting morphisms $w_{i,j} (i \geq j)$ defined by

$$F^01 = 1$$
$$F^{i+1} = F(F^i1) \text{ and } w_{i+1,j+1} = Fw_{i,j} \quad (i \geq j)$$

and for limit ordinals $i$

$$F^i = \lim_{0<i} F^j1 \text{ with the limit cone } w_{i,j}(i > j).$$

This determines an ordinal chain, unique up to natural isomorphism.

If this chain converges at $i$, i.e. the connecting morphism $F^i1 \rightarrow F^{i+1}$ is invertible, then this inverse makes $F^i1$ a coalgebra. This coalgebra is then terminal. See [Ad] where this was first proved in the dual form (initial algebra) and [B] where, independently, the present formulation was used.

2.2. Theorem. [Worrell [W]] Every $\lambda$-accessible endofunctor of Set has a terminal coalgebra obtained in $\lambda + \lambda$ steps. In particular, $\omega + \omega$ steps are sufficient for finitary functors.

2.3. Example. Worrell’s result does not generalize to many-sorted sets: for every cardinal $k$ we can find a finitary functor requiring $k$ steps of the terminal chain.

Indeed, use $k$ sorts and define

$$F : \text{Set}^k \rightarrow \text{Set}^k$$
as follows: for every object $X = (X_i)_{i<k}$ put
\[
(FX)_i = \begin{cases} X_i & \text{if } X_j \neq \emptyset \text{ for some } j < i \\ \emptyset & \text{else} \end{cases}
\]
and define $F$ on morphisms as expected. The terminal chain
\[
(1,1,1,1,\ldots) \leftarrow (\emptyset,1,1,1,\ldots) \leftarrow (\emptyset,\emptyset,1,1,\ldots) \leftarrow \ldots
\]
needs $k$ steps to converge to the initial object, the only (therefore terminal) coalgebra for $F$.

However, $F$ is finitary: let $X = \text{colim} X^t$ be a filtered colimit. We need to prove
\[
(FX)_i = \text{colim}_{t \in T} (FX^t)_i \quad \text{for all } i < k.
\]
This is clear if $(FX)_i = \emptyset$. If not, we have $j < k$ with $\emptyset \neq X_j = \text{colim}_{t \in T} X^t_j$. Therefore, there exists $t_0$ with $X^t_{j_0} \neq \emptyset$. Since $T$ is filtered, we can assume without loss of generality that $t_0$ is initial in $T$, thus, $x^t_{j} \neq \emptyset$ for all $t$. Then $(FX^t)_i = X^t_i$ for all $i$ and the desired equality follows.

2.4. REMARK. Recall that a category $\mathcal{A}$ is called locally presentable, see [GU] or [AR], if there exists an infinite cardinal $\lambda$ such that

(a) $\mathcal{A}$ is complete and cocomplete

and

(b) $\mathcal{A}$ has a set of $\lambda$-presentable objects (i.e., such that their hom-functors are $\lambda$-accessible) whose closure under $\lambda$-filtered colimits is all of $\mathcal{A}$.

2.5. THEOREM. Every accessible endofunctor of a locally presentable category has a terminal coalgebra.

In [MP] a stronger result is proved: if $F$ is accessible, then $\text{Coalg} F$ is locally finitely presentable. Thus, it has a terminal object. This was explicitly used by Barr [B]. In case $F$ preserve monomorphisms, more can be proved:

2.6. THEOREM. Let $\mathcal{A}$ be locally presentable and let $F$ be an accessible endofunctor preserving monomorphisms. Then $F$ has the terminal coalgebra $F^i1$ for some ordinal $i$.

PROOF.

(1) Choose a cardinal $\lambda$ such that $\mathcal{A}$ is a locally $\lambda$-presentable category and $F$ preserves $\lambda$-filtered colimits. Recall from [AR] 1.19 and 1.58, that there exists a representative set $A_\lambda$ of all $\lambda$-presentable objects, and that $\mathcal{A}$ is cowellpowered. Thus, we have a set $\mathcal{A}_\lambda$ of representatives of all quotients of objects in $A_\lambda$. Further, recall that every locally presentable category has a full embedding into $\text{Set}^C$ (for some small category $C$) preserving limits and $\mu$-filtered colimits for some infinite cardinal $\mu$. From now on we consider $\mathcal{A}$ to be a full subcategory of $\text{Set}^C$ closed under limits and $\mu$-filtered colimits. Moreover, the cardinal $\mu$ can be substituted by an arbitrary larger one. We thus assume without loss of generality that

(i) $\mu \geq \lambda$
(ii) \( C \) has less than \( \mu \) morphisms

and

(iii) all objects of \( \overline{A}_\lambda \) are \( \mu \)-presentable in \( \text{Set}^C \).

Condition (ii) implies that an object \( X : C \rightarrow \text{Set} \) of \( \text{Set}^C \) is \( \mu \)-presentable iff the sum of all cardinalities of the sets in the image of \( X \) is less than \( \mu \).

(2) Let \( l_i : L \rightarrow L_i \) \((i < \mu)\) be a limit of a \( \mu^{op} \)-chain in \( \text{Set}^C \). Then for every monomorphism \( m : M \hookrightarrow L \) \( \mu \)-presentable

there exists \( i < \mu \) such that \( l_i \cdot m \) is a monomorphism. In fact, this property clearly holds for limits of \( \mu^{op} \)-chains in \( \text{Set} \). Since \( M \) is \( \mu \)-presentable, for every object \( C \) of \( C \) we have \( \text{card}(M(C)) < \mu \), thus, for the limit \( L(M) \) of \( L_i(M) \) in \( \text{Set} \) there exists an ordinal \( i \) such that \( (l_i)_C \cdot m_C \) is a monomorphism. Due to (ii) above our choice of \( i \) can be made independent of \( C \in C \). Since \((l_i \cdot m)_C \) is a monomorphism for every \( C \), we conclude that \( l_i \cdot m \) is a monomorphism.

(3) We prove that the connecting morphism

\[ w_{\mu+1,\mu} : F^{\mu+1} \rightarrow F^\mu \]

is a monomorphism. Let \( u_1, u_2 : X \rightarrow F^{\mu+1} \) be a pair of morphisms of \( A \) that \( w_{\mu+1,\mu} \) merges, then we prove \( u_1 = u_2 \). Without loss of generality assume \( X \in A_\lambda \) (since \( A_\lambda \) is a generator of \( A \)).

We express \( F^\mu \) as a \( \lambda \)-filtered colimit of \( \lambda \)-presentable objects with a colimit cocone

\[ y_t : Y_t \rightarrow F^\mu \] \((t \in T)\).

Since \( F \) preserves \( \lambda \)-filtered colimits, also

\[ Fy_t : FY_t \rightarrow F^{\mu+1} \] \((t \in T)\)

is a colimit cocone. This colimit is preserved by \( A(X,-) \) because \( X \) is \( \lambda \)-presentable. Consequently, \( u_1, u_2 \) factorize through \( Fy_t \) for some \( t \in T \):

Factorize \( y_t \) as an epimorphism \( e : Y_t \rightarrow \overline{Y}_t \) followed by a strong monomorphism \( m : \overline{Y}_t \rightarrow F^\mu_1 \), see [AR], 1.61. By (iii), the object \( \overline{Y}_t \) is \( \mu \)-presentable in \( \text{Set}^C \). By (2) there exists \( i < \mu \) such that

\[ w_{\mu,i} \cdot m : \overline{Y}_t \rightarrow F^i \] is monic

\[ Fw_{\mu,i} \]
and consequently, since \( F \) preserves monomorphisms,

\[ Fw_{\mu,i} \cdot Fm \text{ is monic.} \]

This last monomorphism merges \( Fe \cdot u'_1 \) and \( Fe \cdot u'_2 \); see the diagram above and recall that \( w_{\mu+1,\mu} \) merges \( u_1 = Fy_t \cdot u'_1 \) and \( u_2 = Fy_t \cdot u'_2 \). This proves

\[ Fe \cdot u'_1 = Fe \cdot u'_2. \]

By composing with \( Fm \) we conclude

\[ u_1 = Fy_t \cdot u'_1 = Fy_t \cdot u'_2 = u_2. \]

(4) All connecting morphisms \( w_{i,\mu} \) with \( i \geq \mu \) are monics. This follows from (3) by easy transfinite induction: recall that \( F \) preserves monics and \( w_{i+1,\mu} = w_{\mu+1,\mu} \cdot Fw_{i,\mu} \), for limit steps use the fact that limits of chains of monics are formed by monics.

Every locally presentable category is wellpowered, see [AR], 1.56. Thus, in the chain of subobjects \( w_{i,\mu} \) of \( F^\mu 1 \) there exists \( i > j \) such that \( w_{i,\mu} \) and \( w_{j,\mu} \) represent the same subobject. From \( w_{i,\mu} = w_{j,\mu} \cdot w_{i,j} \) we conclude that \( w_{i,j} \) is invertible. Thus, so is \( w_{j+1,j} \) (due to \( w_{i,j} = w_{j+1,j} \cdot w_{i,j+1} \)).

\[ \square \]

2.7. Open Problem. Can the assumption that \( F \) preserve monomorphisms be left out in the above theorem?

2.8. Remark. [See [AT2]] If \( A \) is one of the categories sets, many-sorted sets, or vector spaces on a field then all (not necessarily accessible) functors having a terminal coalgebra have a convergent terminal chain. But this result is false e.g. for the category \( A = \text{Set}^\Xi \) of graphs.

3. Relatively Terminal Coalgebras

Throughout this section an endofunctor \( F \) of a category \( A \) is assumed to be given. Recall the category \( \text{Alg} F \) of algebras \( \alpha : FA \rightarrow A \) for \( F \): its morphisms, called algebra homomorphisms, from \( (A,\alpha) \) to \( (B,\beta) \) are morphisms \( f : A \rightarrow B \) in \( A \) with \( f \cdot \alpha = \beta \cdot Ff \). Dually, \( \text{Coalg} F \) has objects \( \alpha : A \rightarrow FA \) and coalgebra homomorphisms from \( (A,\alpha) \) to \( (B,\beta) \) are morphisms \( f \) with \( \alpha \cdot f = Ff \cdot \beta \).

Given an algebra \( \alpha : FA \rightarrow A \) and a coalgebra \( \beta : B \rightarrow FB \), by a coalgebra-to-algebra morphism is meant a morphism \( f : B \rightarrow A \) such that the square

\[
\begin{array}{ccc}
B & \xrightarrow{\beta} & FB \\
\downarrow f & & \downarrow \beta F \\
A & \xleftarrow{\alpha} & FA \\
\end{array}
\]

commutes. A precomposite of \( f \) with a coalgebra homomorphism yields another coalgebra-to-algebra homorphism. The same is true about post-composite \( f \) with an algebra homomorphism.

A fixed point is a (co)algebra \( \alpha : FA \rightarrow A \) with \( \alpha \) invertible. Fixed points form full subcategories both in \( \text{Alg} F \) and \( \text{Coalg} F \).
3.1. Definition. Let \( \alpha : FA \to A \) be an algebra. By a relatively terminal coalgebra is meant a coalgebra \( \hat{\alpha} : \hat{A} \to F\hat{A} \) together with a coalgebra-to-algebra homomorphism

\[
\begin{array}{c}
\hat{A} \xrightarrow{\hat{\alpha}} F\hat{A} \\
\varepsilon \downarrow \quad \downarrow F\varepsilon \\
A \xleftarrow{\alpha} FA
\end{array}
\]

universal in the expected sense: for every coalgebra-to-algebra homomorphism \( h : (B, \beta) \to (A, \alpha) \) there exists a unique coalgebra homomorphism \( \hat{h} : (B, \beta) \to (\hat{A}, \hat{\alpha}) \) with \( h = \varepsilon \cdot \hat{h} \).

3.2. Lemma. For every algebra \( \alpha : FA \to A \) the concept of relatively terminal coalgebra for \( F \) is the same as the concept of terminal coalgebra for the endofunctor \( F\alpha \) of the slice category \( \mathcal{A}/A \) defined by

\[
F\alpha(X \xrightarrow{h} A) = (FX \xrightarrow{Fh} FA \xrightarrow{\alpha} A).
\]

Proof. Let \( \text{Coalg} F/(A, \alpha) \) denote the category of coalgebras over \((A, \alpha)\), i.e., pairs \((B, \beta), h\) consisting of a coalgebra \((B, \beta)\) and a coalgebra-to-algebra homomorphism \( h : B \to A \). Morphisms from \((B, \beta), h\) to \((B', \beta'), h'\) are precisely the coalgebra homomorphisms \( u : B \to B' \) with \( h = h' \cdot u \). By definition the concept of a relatively terminal coalgebra is nothing else than a terminal object of \( \text{Coalg} F/(A, \alpha) \). And this category is isomorphic to the category of coalgebras for \( F\alpha \). Indeed, to give a coalgebra for \( F\alpha \) means to give an object \( h : B \to A \) of \( \mathcal{A}/A \) and a morphism, say \( \beta \), of \( \mathcal{A}/A \):

\[
\begin{array}{c}
B \xrightarrow{\beta} FB \\
\downarrow h \quad \quad \downarrow Fh \\
A \xleftarrow{\alpha} FA
\end{array}
\]

Also morphisms of the two categories above obviously correspond. \( \Box \)

3.3. Corollary. (Lambek’s Lemma) All relatively terminal coalgebras are fixed points of \( F \).

Indeed, by Lambek’s Lemma the terminal coalgebra of \( F\alpha \) is a fixed point of \( F\alpha \), thus, a fixed point of \( F \).
### 3.4. Examples.

(1) Every fixed point $\alpha : FA \twoheadrightarrow A$ has the trivial relatively terminal coalgebra $\alpha^{-1} : A \rightarrow FA$. This follows from the coincidence of coalgebra homomorphisms into this coalgebra and coalgebra-to-algebra homomorphisms into the given algebra.

(2) The relatively terminal colagebras for the trivial algebra $F1 \rightarrow 1$ are precisely the terminal coalgebras for $F$. Thus in contrast to (1), these do not exist in general.

(3) For $F = \text{Id}_{\text{Set}}$ an algebra is a dynamic system given by a set $A$ of states and a next-state function $\alpha : A \rightarrow A$. The relatively terminal coalgebra can be described as the set $\hat{A}$ of all runs $(x_n)_{n \in \mathbb{N}}$ in the dynamic system: here $x_n$ are states such that the next state of $x_{n+1}$ is $x_n$ for every $n$. The coalgebra structure is given by $(x_n) \mapsto (x_{n+1})$ and the universal map by $(x_n) \mapsto x_0$. This follows from Corollary 3.9 below.

(4) For the power-set functor $\mathcal{P}$, coalgebras are the graphs $G$. Given an algebra $\alpha : \mathcal{P}A \rightarrow A$, a coalgebra-to-algebra morphism is a labeling of the vertices of $G$ in $A$,

$$f : G \rightarrow A$$

with the property that the label of every vertex $x \in G$ is $\alpha$ applied to the set of labels of the neighbors of $x$:

$$f(x) = \alpha\{f(y); y \text{ a neighbor of } x\}.$$  

Such labeling is called an $A$-decoration of the graph. (Recall that a decoration, as introduced by Aczel [Ac], is a labeling of vertices by sets such that the label of every vertex is the set of all labels of all neighbors.)

It follows from Corollary 3.3 that no algebra has a relatively terminal coalgebra.

(5) For the finite power-set functor $\mathcal{P}_f$ coalgebras are the finitely branching graphs. For every algebra $\alpha : \mathcal{P}_f A \rightarrow A$ we can describe the relatively terminal coalgebra $\hat{A}$ analogously to the description of the (absolutely) terminal coalgebra due to Barr [B]. Let us call an $A$-labeled tree extensional if no node has two isomorphic maximal sub-trees (w.r.t. isomorphisms respecting the labels). Every $A$-labeled tree has a unique extensional quotient, obtained by recursively identifying syblings defining isomorphic subtrees. We call two $A$-labeled trees $t$ and $s$ Barr-equivalent if for every $n \in \mathbb{N}$ the cuttings of $t$ and $s$ at level $n$ have the same extensional quotient.

Let $\hat{A}$ be the coalgebra of all finitely branching $A$-decorated trees modulo Barr equivalence. This is a coalgebra of $\mathcal{P}_f$: to every tree assign the set of all children of the root. And $\hat{A}$ has an $A$-decoration given by the label of the root. The proof that $\hat{A}$ is relatively terminal is analogous to the proof for $A = 1$ (that all finitely branching trees modulo the Barr equivalence are terminal for $\mathcal{P}_f$) in [B].

### 3.5. Proposition. Let $\mathcal{A}$ be a category in which subobjects of any object form a complete lattice. Given a monomorphisms-preserving endofunctor, then every "pre-fixed point", i.e., monomorphism $\alpha : FA \rightarrow A$, has a relatively terminal coalgebra.

**Remark.** We will prove that the algebra $\alpha : FA \rightarrow A$ has a greatest subalgebra which is a fixed point, and the inverse yields the relatively terminal coalgebra. The universal arrow $\varepsilon : A \rightarrow \hat{A}$ is proved to be a monomorphism.
Proof. We have a function $\varphi$ on the complete lattice of all subobjects of $A$ assigning to a subobject $m : M \rightarrow A$ the subobject $\alpha \cdot Fm : FM \rightarrow A$. Since $\varphi$ is monotone, by Knaster-Tarski fixed-point theorem [Ta] $\varphi$ has a greatest fixed point. Now to be a fixed point of $\varphi$ means precisely to be a subalgebra with an invertible structure morphism:

\[
\begin{array}{ccc}
FM - \sim & \rightarrow & M \\
Fm \downarrow & \varphi(m) \downarrow & m \\
FA & \alpha & \rightarrow A
\end{array}
\]

Let $\varepsilon : \hat{A} \rightarrow A$ be the greatest fixed point of $\varphi$, and let $\hat{\alpha} : \hat{A} \rightarrow F\hat{A}$ be the corresponding isomorphism. Then $\varepsilon$ is universal because every coalgebra-to-algebra homomorphism

\[
\begin{array}{ccc}
B & \beta & \rightarrow FB \\
\downarrow h & \downarrow Fh & \\
A & \alpha & \leftarrow FA
\end{array}
\]

factorizes through $\varepsilon$. (To verify this, it is only needed to prove that whenever $h$ factorizes through a subobject $m : M \rightarrow A$, then it factorizes through $\varphi(m)$. Indeed, from $h = m \cdot k$ derive $h = \alpha \cdot F(m \cdot k) \cdot \beta = \varphi(m) \cdot Fk \cdot \beta$.) Thus we have a factorization $\hat{h} : B \rightarrow \hat{A}$ with $\varepsilon \cdot \hat{h} = h$, and then $\hat{h}$ is a coalgebra homomorphism: to verify $\hat{\alpha} \cdot \hat{h} = F\hat{h} \cdot \beta : B \rightarrow F\hat{A}$ we use that $\alpha \cdot F\varepsilon$ is a monomorphism with

$\alpha \cdot F\varepsilon \cdot (\hat{\alpha} \cdot \hat{h}) = \varepsilon \cdot \hat{h} = h = \alpha \cdot Fh \cdot \beta = \alpha \cdot F\varepsilon \cdot (F\hat{h} \cdot \beta)$.

\[\Box\]

3.6. Corollary. Every relatively terminal coalgebra yields a coreflection of the given algebra in the full subcategory of $\text{Alg}F$ formed by all fixed points. That is:

(a) $\varepsilon$ is an algebra homomorphism

\[
\begin{array}{ccc}
F\hat{A} & \hat{\alpha} \rightarrow & \hat{A} \\
\downarrow F\varepsilon & \downarrow \varepsilon & \\
FA & \alpha & \rightarrow A
\end{array}
\]

(b) every fixed point $\beta : FB \rightarrow B$ has the property that all algebra homomorphisms into $A$ uniquely factorize through $\varepsilon$.

3.7. Example. Unfortunately, we cannot define the relatively terminal coalgebras as coreflections in the subcategory of fixed points. Consider the modified power-set functor $P'$ sending $\emptyset$ to $\emptyset$ and all nonempty sets $X$ to $PX$: it has almost no relatively terminal coalgebras, see Example 3.13. However, since $\emptyset$ is its only fixed point, this is the coreflection of every algebra in the subcategory of fixed points.

3.8. A Limit Construction. Recall from Notation 2.1 that a terminal coalgebra of $F$ is obtained as a limit of the $\omega^{op}$-chain

\[1 \leftarrow F1 \leftarrow FF1 \leftarrow \ldots\]

whenever $F$ preserves this limit. Applied to $F\alpha$ of Lemma 3.2 this is the $\omega^{op}$-chain obtained by iterating $F$ on $A \leftarrow FA$:
3.9. **Corollary.** If $F$ preserves limits of $\omega^{op}$-chains, then every algebra $\alpha : FA \to A$ has a relatively terminal coalgebra which is the limit $F^\omega A$ of the $\omega^{op}$-chain

$$A \leftarrow F\alpha \leftarrow FFA \leftarrow \ldots$$

(1)

3.10. **Example.**

(a) Let $\Sigma$ be a (possibly infinitary) signature. Then $\Sigma$-algebras are algebras for the polynomial functor $H_\Sigma X = \coprod_{\sigma \in \Sigma} X^n$ where $n$ is the arity of $\sigma$. This functor preserves limits of $\omega^{op}$-chains.

Given a $\Sigma$-algebra $A$, then $\widehat{A} = \lim F^n A$ can be described as the set of all trees (up to isomorphism) labelled in $\Sigma \times A$ with the following property: given a node with label $(\sigma, a)$ where $\sigma$ is $n$-ary, this node has precisely $n$ children, and their labels $(\sigma_i, a_i)$ for $i < n$ satisfy

$$a = \sigma^A (a_i)_{i < n}$$

(b) CPO-enriched categories. As mentioned in the introduction, Dana Scott constructed in [S] a model of $\lambda$-calculus as a relatively terminal coalgebra for an endofunctor of the category of continuous lattices. Later Gordon Plotkin and Mike Smyth proved that the same procedure works in every category enriched over $\omega$CPO, see [SP], which means that the hom-sets $A(X, Y)$ carry an $\omega$CPO structure (i.e., a poset with a least element and joins of $\omega$-chains) and composition is strict and continuous (i.e., preserves least element and $\omega$-joins). We also assume that $A$ has limits of $\omega^{op}$-sequences which are $\omega$CPO-enriched. An endofunctor $F$ is called *locally continuous* provided that the induced maps from $A(X, Y)$ to $A(FX, FY)$ are continuous, i.e., $F(\bigsqcup f_n) = \bigsqcup Ff_n$ for every $\omega$-chain $(f_n)$ in $A(X, Y)$.

For every algebra $\alpha : FA \to A$ for which $\alpha$ is a projection, i.e., there exists $e : A \to FA$ with $\alpha \cdot e = id$ and $e \cdot \alpha \sqsubseteq id$, the limit $F^\omega A$ is the relatively terminal coalgebra. This follows from the coincidence of the limit of the chain (1) and the colimit of the $\omega$-chain $A \xrightarrow{e} FA \xrightarrow{F\alpha} FFA \xrightarrow{F\alpha} \ldots$ as established on [SP].

3.11. **Remark.** We know from Proposition 3.5 that if $\alpha : FA \to A$ is a monomorphism, then the relatively terminal coalgebra exists and $\varepsilon : \widehat{A} \to A$ is a monomorphism. Now if $\alpha : FA \to A$ is a split epimorphism, then so is $\varepsilon : \widehat{A} \to A$.

3.12. **Lemma.** Let $\alpha : FA \to A$ be a split epimorphism. If a relatively terminal coalgebra exists, then $\varepsilon : \widehat{A} \to A$ is also a split epimorphism.

**Proof.** Given

$$\alpha \cdot m = id$$

for some $m : A \to FA$

we obtain a trivial coalgebra-to-algebra morphism

$$\begin{array}{ccc}
A & \xrightarrow{m} & FA \\
\downarrow{id} & & \downarrow{Fid} \\
A & \xleftarrow{\alpha} & FA
\end{array}$$

Then $\varepsilon \cdot \widehat{id} = id$ gives the desired splitting. \qed
3.13. Example. For the functor $P'$ of Example 3.7 no surjective algebra $\alpha : P'A \to A$ with $A \neq \emptyset$ has a relatively terminal coalgebra. Indeed, no fixed point $\hat{A}$ of $P'$ has a surjective map $\epsilon : \hat{A} \to A$.


(a) The limit of the chain (1) is denoted by $F^\omega A$ with limit projections

$$\hat{a}_i : F^\omega A \to F^i A \quad (i < \omega).$$

Then $F^\omega A$ is an algebra: we have the unique algebra structure

$$\overline{\alpha} : F(F^\omega A) \to F^\omega A$$

with

$$\hat{a}_0 \cdot \overline{\alpha} = \alpha \cdot F\hat{a}_0 \quad \text{and} \quad \hat{a}_{i+1} \cdot \overline{\alpha} = F\hat{a}_i. \quad (2)$$

(b) For every coalgebra-to-algebra homomorphism

$$B \xrightarrow{\beta} FB \xrightarrow{Fh} FA \xleftarrow{\alpha} A$$

we obtain a cone $h_i : B \to F^i A$ of the chain (1) by

$$h_0 = h \quad \text{and} \quad h_{i+1} = Fi \cdot \beta. \quad (3)$$

The unique factorization is denoted by

$$\overline{h} : B \to F^\omega A.$$

3.15. Proposition. The relatively terminal coalgebras for the three algebras

$$FA \xrightarrow{\alpha} A, \quad FFA \xrightarrow{F\alpha} FA \quad \text{and} \quad F(F^\omega A) \xrightarrow{\overline{\alpha}} F^\omega A$$

are the same.

Proof.

(a) The category $\text{Coalg } F/(A, \alpha)$ of coalgebras over $(A, \alpha)$, see Lemma 3.2, is isomorphic to the category of coalgebras over $(FA, F\alpha)$. Indeed, we have a functor

$$V : \text{Coalg } F/(A, \alpha) \to \text{Coalg } F/(FA, F\alpha)$$

which is defined on objects by

$$B \xrightarrow{\beta} FB \xrightarrow{Fh} FFB \xleftarrow{Fh} FFA$$

$$\xrightarrow{\beta} FB \xrightarrow{Fh} FFB \xleftarrow{Fh} FFA$$
and on morphisms by \( Vu = u \). This is inverse to the functor defined on objects by

\[
\begin{array}{c}
\begin{array}{ccc}
B & \xrightarrow{\beta} & FB \\
\downarrow h & & \downarrow Fh \\
FA & \xleftarrow{F\alpha} & FFA
\end{array} & \begin{array}{c}
A \leftarrow \gamma \\
\downarrow F\alpha \\
A \leftarrow \gamma
\end{array}
\end{array}
\]

(b) The category of coalgebras over \((A, \alpha)\) is also isomorphic to the category of coalgebras over \((F^\omega A, \overline{\alpha})\). In the notation 3.14 we have a functor

\[ \overline{V} : \text{Coalg } F/(A, \alpha) \rightarrow \text{Coalg } F/(F^\omega A, \overline{\alpha}) \]

defined on objects by

\[
\begin{array}{c}
\begin{array}{ccc}
B & \xrightarrow{\beta} & FB \\
\downarrow h & & \downarrow Fh \\
F^\omega A & \xleftarrow{F\alpha} & F(F^\omega A)
\end{array} & \begin{array}{c}
A \leftarrow \gamma \\
\downarrow F\alpha \\
A \leftarrow \gamma
\end{array}
\end{array}
\]

The right-hand square commutes since for every \( i < \omega \) we have, due to (2) and (3),

\[ \hat{a}_{i+1} \cdot (\overline{\alpha} \cdot F\overline{h} \cdot \beta) = F(\hat{a}_i \cdot \overline{h}) \cdot \beta = Fh_i \cdot \beta = \hat{a}_{i+1} \cdot \overline{h} \]

The inverse functor is given on objects by

\[
\begin{array}{c}
\begin{array}{ccc}
B & \xrightarrow{\beta} & FB \\
\downarrow h & & \downarrow Fh \\
F^\omega A & \xleftarrow{\alpha \cdot \gamma} & F(F^\omega A)
\end{array} & \begin{array}{c}
A \leftarrow \gamma \\
\downarrow F\alpha \\
A \leftarrow \gamma
\end{array}
\end{array}
\]

\[\square\]

4. Finitary Functors

Recall that an endofunctor \( F \) is called \textit{finitary} if it preserves filtered colimits.

4.1. \textbf{Remark}.

(a) For set functors this means that given an arbitrary element \( x \in FX \), there exists a finite subset \( u : U \hookrightarrow X \) such that \( x \) lies in the image of \( Fu \).

(b) For example \( \text{Id} \) and \( \mathcal{P}_f \) are finitary, and \( \mathcal{P} \) is not. The functor \( H_\Sigma \) (Example 3.10) is finitary iff \( \Sigma \) is a finitary signature.
(c) James Worrell proved in [W] that finitary set functors $F$ have terminal coalgebras obtained in $\omega + \omega$ steps of the iterative construction. We now prove that, analogously, every algebra $\alpha : FA \rightarrow A$ has a relatively terminal coalgebra obtained in $\omega + \omega$ steps: the first $\omega$ steps yield the algebra $\overline{\alpha} : F(F^\omega A) \rightarrow F^\omega A$ of Notation (3.14), the next $\omega$ steps:

$$F^\omega A \xleftarrow{\overline{\alpha}} F(F^\omega A) \xleftarrow{F \overline{\alpha}} FF(F^\omega A) \xleftarrow{FF \overline{\alpha}} \ldots$$

(4)

are just the same construction applied to that new algebra. It turns out that $\overline{\alpha}$ is always a monomorphism so that the next limit is the intersection of the chain (4) of subobjects. And $F$ preserves this intersection, consequently, $F^\omega(F^\omega A)$ is a relatively terminal coalgebra for $F^\omega A$ or, equivalently, for $A$: see Proposition 3.15.

(d) Our proof uses Worrell’s idea, but is slightly simpler. Observe that we cannot apply Lemma 3.2 here, because $F_\alpha$ works on the category $\text{Set}/A$ of $A$-sorted sets, and we know that Worrell’s result does not extend to many sorted sets (see Example 2.3).

4.2. Lemma. For every finitary set functor $F$ there exists a finitary set functor $F'$ preserving intersections and agreeing with $F$ on all nonempty sets (and functions).

Proof. The existence of a functor $F'$ such that $F$ preserves finite intersections and agrees with $F$ on all nonempty sets is established in [Tr]. If $F$ is finitary, so is $F'$. To prove that $F'$ preserves all intersections, let $m = \bigcap_{i \in I} m_i$ be an intersection of subsets $m_i : M_i \rightarrow X(i \in I)$. For every element $x \in FM$ lying in the image of $Fm_i$ for all $i \in I$ we are to prove that $x$ lies in the image of $m$. Choose $u$ in Remark 4.1 with $U$ of the minimal cardinality. Since $F$ preserves the intersection $u \cap m_i$ for every $i \in I$, the minimality of $U$ implies $u \subseteq m_i$. Thus, $u \subseteq m$, consequently, $x$ lies in the image of $Fm$. \[\square\]

4.3. Theorem. For every algebra $\alpha : FA \rightarrow A$ of a finitary set functor $F$ the relatively terminal coalgebra is the limit of the $\omega^\omega$-chain

$$F^\omega A \xleftarrow{\overline{\alpha}} F(F^\omega A) \xleftarrow{F \overline{\alpha}} FF(F^\omega A) \xleftarrow{FF \overline{\alpha}} \ldots$$

Remark. We will see that $\overline{\alpha}$ is monic; thus, the limit is an intersection.

Proof.

(1) Let $F$ preserve intersections. It is sufficient to prove that $\overline{\alpha}$ is monic: then $F$ preserves the limit (=intersection) of the above chain. Given $x, y \in F(F^\omega A)$ there exists a finite subset $m : M \rightarrow A^\omega$ with $x, y$ in the image of $Fm$, see Remark 4.1. The limit cone $(\overline{a}_i)_{i \in \omega}$ of Notation 3.14 fulfills, since $M$ is finite: there exists $j$ with $\overline{a}_j \cdot m : M \rightarrow F^j A$ monic. Thus $F\overline{a}_j \cdot Fm$ is monic, hence, $x \neq y$ implies $F\overline{a}_j(x) \neq F\overline{a}_j(y)$. Since $\overline{a}_{j+1} \cdot \overline{\alpha} = F\overline{\alpha}_j$, see (2), we conclude $\overline{\alpha}(x) \neq \overline{\alpha}(y)$, as required.

(2) Let $F$ be arbitrarily and let $F'$ be the functor of Lemma 4.2. Every algebra $\alpha : FA \rightarrow A$ with $A \neq \emptyset$ is also an algebra for $F'$. And the relatively terminal coalgebra for $F'$ is clearly relatively terminal for $F$ too. If $A = \emptyset$, then $FA = \emptyset$, thus, $\alpha = \text{id}_\emptyset$ and the theorem holds trivially.

\[\square\]
5. Relatively terminal chain

Let $\alpha : FA \rightarrow A$ be an algebra for an endofunctor of a complete category. The *relatively terminal chain* is the terminal chain of the endofunctor $F_\alpha$ of Lemma 3.2. Explicitly: it has objects $F^iA$ ($i \in \text{Ord}$) and connecting morphisms $\alpha_{i,j} : F^iA \rightarrow F^jA$ ($i \geq j$) defined by transfinite induction as follows:

\[
F^0A = A, \quad F^1A = FA \quad \text{and} \quad \alpha_{10} = \alpha \\
F^{i+1}A = F^i(F^iA) \quad \text{and} \quad \alpha_{i+1,j+1} = F\alpha_{i,j}
\]

and for limit ordinals $i$

\[
F^iA = \lim_{j<i} F^jA \quad \text{with the limit cone} \quad \alpha_{i,j}(i > j).
\]

Recall that the slice category $\mathcal{A}/A$ has all colimits and all connected limits computed as in $\mathcal{A}$. Therefore, every accessible endofunctor $F$ yields an accessible endofunctors $F_\alpha$. And if $F$ preserves monomorphisms, so does $F_\alpha$. Thus Theorems 2.5 and 2.6 yield

5.1. COROLLARY. Every accessible endofunctor $F$ of a locally presentable category has relatively terminal coalgebras for all algebras.

If $F$ preserves monomorphisms, then every algebra $A$ has a relatively terminal coalgebra $\hat{A} = F^iA$ for some ordinal $i$.

5.2. REMARK. In case of endofunctors of $\text{Set}$ we can say more: whenever $F$ (not necessarily accessible) has, for a given algebra $\alpha : FA \rightarrow A$, a relatively terminal coalgebra, then $\hat{A} = F^iA$ for some ordinal $i$. Indeed, use Lemma 3.2: since $F_\alpha$ is an endofunctor of the category $\text{Set}/A$ of $A$-sorted sets, we can apply the result of Remark 2.8: all terminal coalgebras in many-sorted sets are obtained by the terminal chain.

References


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